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Iterative Methods for Solving Economic Models

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ITERATIVE METHODS FOR THE SOLUTION OF ECONOMIC MODELS *

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I. THE MATHEMATICS

Before the introduction of digital computers, it was necessary to solve economic models by hand. This imposed a severe restriction on the size and type of model which could be solved. Although there are numerous well established methods for solving systems of linear equations, the amount of work required for solution increases so rapidly as the size of the system increases that hand computation soon becomes impracticable. Also to restrain the equations of the system to be linear severely restricts the validity of an economic model. Since it is extremely difficult to solve non-linear systems by hand, it was only the small, linear, economic models which could be solved before digital computers became available.

The powerful computational abilities of the digital computer, in effect, removed the restriction of size from model building. Using the traditional methods, it became possible to solve linear systems of an extremely high order. In order for a computer to function it requires as algorithm, and these traditional methods were easily translated into computer algorithms. However, even with the use of a computer, the solution of a non-linear system of equations is still quite difficult. This section will attempt to establish the mathematical foundation for several of the algorithms which are available for the solution of such systems. Since almost all of the algorithms available are extensions of methods for the solution of linear systems, the mathematics of these algorithms has been established in relation to their application to linear systems.

Consider the general problem of finding the solution to a set of linear algebraic equations. Symbolically this involves finding a vector

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*This working paper represents part of an undergraduate MIT thesis submitted to the Department of Economics in June, 1967. Details on the actual numerical computations can be found there.

X which satisfies the equality

$$AX = B \qquad (1)$$

where A is a non-singular square matrix and B is a vector. The practical methods for finding such a solution can be divided into two large classes, namely, the exact methods and the methods of successive approximations. In the first class are the simple Gaussian eliminations techniques which are taught in high school algebra classes. Also in this class are the matrix inversion methods which involve solving equation (1) for X

$$X = A^{-1}B$$

It is characteristic of the exact methods that they produce the exact solution in a given number of operations which is a function solely of the order of the system of equations and not of the matrix of coefficients A.

The exact methods have several disadvantages when applied to relatively large systems of equations. With these methods the number of operations required for solution is roughly proportional to the cube of the number of equations in the system: thus as the size of the model increases the effectiveness of these methods rapidly decreases. Because of the limited length of the computer registers, there is a certain amount of rounding error introduced with every operation. With the exact methods these errors are cumulative, and there is the possibility of introducing significant error. In general economic models are too large to be effectively handled by the exact methods: non-linear economic models which contained more than ten equations could be more efficiently solved by other methods.

Economic models also have the characteristic that any given equation will contain relatively few variables. For a linear model this would imply

that the matrix of coefficients, A, is sparse i.e. contains many zeros. The exact methods are not able to take advantage of this and the number of operations required for solution is not reduced.

The second class of methods, the methods of successive relaxation, have none of the obvious disadvantages of the exact methods, and therefore appear to be better suited for the solution of economic models. The rest of this section will be devoted to establishing the mathematical basis for this class of methods.

The general nature of the methods of successive relaxation is contained in the equation

$$X^{r+1} = A^{(r)} + f(X^{(r)}) \qquad (2)$$

where X is a single number, a function, a vector, or a matrix, according to context. $A^{(r)}$ is independent of X but not necessarily of r. The solution is the limit of the series of estimates, X^{r+1}, as r goes to infinity. For the solution of a system of linear equations the original vector is immaterial: if the process converges, it will converge from any point. Iterative methods form an important subset of the methods of successive relaxation, and because their mathematics is relatively straightforward, they serve as a good introduction to this area.

Let the system of linear equations be given in the following form

$$x_1 = a_{11}x_1 + a_{12}x_2 \ldots\ldots + a_{1n}x_n + c_1$$

$$- - - - - - - - - - - - - - - - - - - -$$

$$x_n = a_{n1}x_1 + a_{n2}x_2 \ldots\ldots + a_{nn}x_n + c_n$$

At first glance such a representation appears somewhat artificial because of the presence of like terms on both sides of the equality sign. However, this arrangement of a linear system is the basis for an iterative process

which, although impractical for applications, is very convenient for illustrating certain properties of the iterative methods.

Let us now write system (1) in the form

$$X = AX + C$$

where A is a matrix of coefficients and C is a vector of constant terms. Starting with an arbitary initial vector $X^{(0)}$, the following series is formed

$$X^{(1)} = AX^{(0)} + C$$

$$X^{(2)} = AX^{(1)} + C$$

- - - - - - - - -

$$X^{(k)} = AX^{(k-1)} + C$$

If the sequence $X^{(0)}, X^{(1)}, \ldots$ has a limit X, this limit will be the solution to system (2), for X^{r} will equal X^{r+1} and this will produce the equality $X = AX + C$. Following through this recursive function it is easily verified that

$$X^{(k)} = A^{k}X^{(0)} + (I + A + \ldots A^{k-1})C \qquad (3)$$

Now for this process to converge it is necessary and sufficient that the series

$$(I + A + \ldots A^{k-1})C \qquad (4)$$

also converges. It can be proven that this series converges if and only if all of the eigenvalues of the matrix A are of absolute value less than one. And it is a necessary and sufficient condition for the convergence of all iterative processes that the eigenvalues of the iterative matrix be of absolute value less than one.

For this method the solution vector is the sum of a series of terms, and referring back to the original mathematical description of the methods

of successive approximations, X in this case is a function. The closer the largest eigenvalue is to zero in absolute value, the faster the terms in the series in (4) tend to zero and the faster is convergence. After a few iterations one should be able to estimate the speed with which this sum is approaching a limit and hence estimate the limit. Since the speed of convergence is solely a function of the largest eigenvalues of the iteration matrix, it is possible to estimate the solution at every iteration knowing only the value of the largest eigenvalue.

Although this method is not very practicable for application to real systems, it shows more clearly than other methods that both the criterion for convergence and the speed of convergence of the iterative methods are a function of the largest eigenvalue of the iteration matrix.

The Jacobi iterative method is a more efficient method, particularly when adopted to the solution of non-linear systems. Let a system of linear algebra equations be represented in the matrix notation

$$A\underline{X} = \underline{B}. \tag{5}$$

In order to express exactly the mathematics of the Jacobi method, it is necessary to decompose the matrix of coefficients, A, into three components, U, L, and D, which represent the upper-triangular, lower-triangular, and diagonal elements of A.[1]Using this notation the system now becomes

$$(U + L + D)\underline{X} = \underline{B}.$$

Taking advantage of the commutative property of matrix multiplication, this equation can be transformed into

$$D\underline{X} = \underline{B} - (U + L)\underline{X}.$$

The Jacobi method approximates this equality by the following iteration

function

$$D\underline{X}^{r+1} = \underline{B} - (U + L)\underline{X}^{r}$$

or

$$X^{r+1} = D^{-1}\,\underline{B} - D^{-1}(U + L)\underline{X}^{r}$$

If this process converges, then for sufficiently large $\underline{r}$, X^{r+1} will not differ significantly from X^{r}, and by working backwards it can be easily verified that this vector is the solution of the original system. To see what this process actually involves, consider the following representation of a linear system

$$l_1 = a_{11}x_1 + a_{12}x_2 \ldots a_{1n}x_n$$
$$- - - - - - - - - - - - - -$$
$$l_n = a_{n1}x_1 + a_{n2}x_2 \ldots a_{nn}x_n$$

Using this notation the approximation to the solution on the (r+1)th iteration of the Jacobi method is

$$x_1^{r+1} = \frac{a_{12}x_2^{2}}{a_{11}} + \frac{a_{13}x_3^{2}}{a_{11}} \ldots - \frac{1}{a_{11}}$$

$$- - - - - - - - - - - - - - - - - -$$

$$x_n^{r+1} = \frac{a_{n1}x_1^{r}}{a_{nn}} + \frac{a_{n2}x_2^{r}}{a_{nn}} \ldots - \frac{l_n}{a_{nn}}$$

For the Jacobi scheme the matrix of iteration is $D^{-1}(U + L)$, and this process will converge to the solution vector if all of the eigenvalues of this matrix are less than one in magnitude.

The Gauss-Seidel and the Jacobi iterative are closely related. Decomposing the matrix of coefficients in the same manner, equation (5) is solved slightly differently to produce the Gauss-Seidel iterative method

$$\underline{X}^{r+1} = (I-L)^{-1}(D + U)\,\underline{X}^{r} + (I - L)^{-1}\underline{B}$$

Using the more detailed notation, the solution vector on the (r+1)th iteration is approximated by the vector

$$x_1^{r+1} = \frac{a_{12}x_2^{r}}{a_{11}} + \frac{a_{13}x_3^{r}}{a_{11}} \cdots - \frac{l_1}{a_1},$$

$$x_2^{r+1} = \frac{a_{21}x_1^{r+1}}{a_{22}} + \frac{a_{23}x_3^{r}}{a_{22}} \cdots \frac{-l_2}{a_{22}}$$

- - - - - - - - - - - - - - - - -

$$x_n^{r+1} = \frac{a_{n1}x_2^{r+1}}{a_{nn}} + \frac{a_{n2}x_2^{r+1}}{a_{nn}} \cdots\cdots \frac{-l_n}{a_{nn}}$$

This process differs from the Jacobi method in that the vector of approximations, $\underline{X}$, is immediately updated with the new estimates of the individual components. Thus for a system of order $\underline{n}$, the solution vector is updated $\underline{n}$ times during every iteration, whereas for the Jacobi method it is updated only once for every iteration.

The matrix of iteration for the Gauss-Seidel method is

$$(I - L)^{-1}(D + U)$$

and again it is necessary that all of the eigenvalues of this matrix be of absolute value less than one for the process to converge. For both of these methods there are many ways in which the system of equations can be normalized. The term normalization in this context refers to which variable a given equation is solved for. For instance, the first equation may be solved for the tenth variable. If the matrix of coefficients for a $\underline{n}$ order linear system contained no zeros, there would be $\underline{n}$ factorial different normalizations possible. Since each normalization produces a different matrix of iteration, there is the possibility that for a given system the process will converge with certain normalizations and not for others. Also since the solution vector in the Gauss-Seidel method is continually

updated, the way in which the equations are ordered will affect the matrix of iterations. For the Jacobi method the ordering is immaterial.

There is an extensive list of conditions, which if satisfied, guarantee that the eigenvalues of the matrix are less than one in absolute value. Unfortunately these are only sufficient conditions and there is a large class of matrices for which the eigenvalues are less than one in absolute value which do not satisfy these conditions. Since it requires approximately the same amount of work to calculate the largest eigenvalue as to perform the entire iteration process, the determination of the largest eigenvalue is an impractical way of testing for convergence. Unfortunately, as yet, there is no practical algorithm for normalization or ordering, and one can only proceed on a hit or miss basis.

As was seen, in solving a system of linear equations given in the form $\underline{X} = \underline{A}\underline{X} + \underline{F}$, the solution vector is the sum of a series of vectors. Since the speed with which this sum converges to the solution vector is a function of the largest eigenvalue in absolute value of the iteration matrix, it is possible at every step to estimate the final sum of this series. And for every iteration scheme, the speed of convergence is inversely proportional to the largest eigenvalue of the iteration matrix. This implies that when the largest eigenvalue is close to one in absolute, there is only a small step taken towards the solution vector during each iteration. Thus the speed of convergence can be improved by taking a larger step during each iteration than the iterative scheme prescribes, and the closer the largest eigenvalue is to one in absolute value, the larger the step should be. For every iteration method the following approximation holds

$$X \simeq X^{(K)} + \frac{X^{(K+1)} - X^{(K)}}{1 - e_1}$$

where e_1 is the largest eigenvalue in absolute value of the iteration matrix. This approximation can then be taken as the current estimate of the solution for the next iteration. Thus the number of steps required for solution is significantly reduced when the value of the largest eigenvalue is known.

Besides the iterative schemes, there is another class of methods which come under the heading of methods of successive approximation. These are known as the methods of steepest descent, or the gradient methods. For an understanding of these methods, consider the relationship which results when a vector of initial estimates is substituted into a system of linear equations

$$R = A\underline{X} - \underline{B}.$$

If $\underline{X}$ were the true solution vector, then the vector $\underline{R}$ would have only zeros for entries. The purpose of the gradient method is to produce a series of vectors, the limit of which forces $\underline{R}$ into the null vector. During each iteration it changes a given variable such that the sum of squares of the components of the resulting vector $\underline{R}$ is brought to a minimum. For instance, if the residual of the first equation is 100, then by changing variable one in this equation by

$$\frac{100}{a_{11}}$$

units, the residual of this equation will become zero. The general gradient method reduces the sum of squared residuals by taking the weighted average of these changes for each variable. The change in variable $\underline{i}$ during iteration is given by

$$\frac{a_{i1}r_1 + a_{i2}r_2 \ldots . + a_{in}r_n}{a_{i1}^2 + a_{i2}^2 \ldots \ldots . + a_{in}^2}$$

It can be proven that for a system of linear equations the gradient method will converge from any starting point; however the computation for every iteration is lengthy and the rate of convergence may often be slow. The Jacobi and the Gauss-Seidel method have the advantage that convergence can be very rapid, but these processes may not converge.

The application of these methods to non-linear systems is relatively straightforward. During each iteration the partial derivatives of each variable are held constant; this suppresses the non-linearities and the system of equations becomes linear. For example consider the following equation

$$10 = 5x + wyz.$$

If the current estimates for the solution values of w, x, y, and z, are respectfully 2, 4, 6, and 8, this equation would be forced into one of the following forms

$$10 = 5x + 12z$$

$$10 = 5x + 16y$$

$$10 = 5x + 48w$$

When the non-linear equations are linearized in this manner, it is possible to apply the algorithms which were described in this chapter. There is a great deal of mathematics concerned with the methods of successive approximations as applied to linear systems of equations, but unfortunately there has been little written about these methods as applied to non-linear systems. To a great extent this is due to the fact that the class of non-linearities is so large that there is little which can be said about the mathematics of such systems. Economic models contain a restricted class of non-linearities. The object of this paper was to apply several of these iterative methods, particularly the Gauss-Seidel, to an economic model to see how efficiently they produced the solution, and if there were any characteristics in their application to economic models.

II. THE MODEL

It can be seen by solving some ordinary systems of linear equations that when the Gauss-Seidel method converges, it converges very quickly. However, just as the mathematics implies, this method does not always produce convergence. On the other hand, the gradient method always converges, but the rate at which it approaches the solution is generally slow. The application of these two methods to non-linear economic models raises some interesting questions. In what portion of the time will the Gauss-Seidel method converge, and how fast is the average rate of convergence? What is the best way to solve for the variables and then order the equation in order to produce the optimum conditions for convergence of the Gauss-Seidel method? To what extent is convergence conditioned upon the starting values? Will the gradient method always converge when applied to a non-linear system which has a real solution?

In an attempt to answer these questions, these methods were used to solve a real economic model. The model which was chosen was a modification of the "Revised Klein-Goldberger Model."[3] This is a macro-econometric model which presents a Keynesian analysis of the United States economy. The equations for this model were estimated from annual time series of the United States economy for the period 1929 to 1962, with the years 1942 to 1945 omitted. These iterative methods were used on this model to solve a simulation of the 1929 economy. Originally the model contained twenty-seven equations, but when used to simulate the 1929 economy, four of these equations become recursive. Of the remaining twenty-three equations twelve are non-linear. With one exception, all of the non-linear

terms are products which have the variable representing "price" as one of the terms. This makes the system non-linear enough to give the methods a good test. The model is large enough to place a premium on speed. The difference of a half a second between two methods when solving a system of five equations may not be important, but as the size of the system increases the difference between the two methods is likely to increase more than proportionally. But perhaps the most important characteristic of the model is that it is a good representation of modern Keynesian economics; it is a representative model.

The model contains the following variables:

1	C_d	Consumption of durables, billions of 1954 dollars.
2	C_n	Consumption of non-durables and services, billions of 1954 dollars.
3	R	Residential construction, billions of 1954 dollars.
4	H	Stock of inventories, billions of 1954 dollars.
5	I_m	Imports, billions of 1954 dollars.
6	h	Index of hours worked per week, 1954 = 1.00.
7	w	Annual earnings, thousands of dollars.
8	W	Wages and salaries and supplements to wages and salaries, billions of 1954 dollars.
9	N_w	Wage and salary workers, millions.
10	S_c	Corporate saving, billions of 1954 dollars.
11	P_c	Corporate profits, billions of 1954 dollars.
12	Π_r	Rental income and net interest, billions of 1954 dollars.
13	X	Gross national product, billions of 1954 dollars.
14	Y	Personal disposable income, billions of 1954 dollars.
15	Π	Proprietors' income, billions of 1954 dollars plus P_c and IVA.

16	IT	Indirect taxes, billions of current dollars.
17	T_c	Corporate profits taxes, billions of current dollars.
18	PT	Personal taxes, billions of current dollars.
19	SI	Contributions for social insurance, billions of current dollars.
20	BT	Business transfers, billions of current dollars.
21	GT	Government transfers, billions of current dollars.
22	p	Implicit GNP deflator, 1954 = 1.00.
23	IVA	Inventory valuations adjustment, billions of current dollars.
24	I	Investment in plant and equipment, billions of 1954 dollars.
25	r_s	Yield on prime commercial paper, 4-6 months, per cent.
26	D	Capital consumption allowances, billions of current dollars.
27	r	Average yield on corporate bonds (Moody's), per cent.

Exogenous Variables

E1	ID	Net interest paid by government, billions of current dollars.
E2	W_g	Government wages and salaries, billions of 1954 dollars.
E3	p_m	Implicit price deflator for imports, 1954 = 1.00.
E4	N_g	Government employees, millions.
E5	N_s	Self-employed workers, millions.
E6	N_L	Total labor force, millions.
E7	GS	Subsides - current surplus of government enterprise, billions of current dollars.
E8	D_u	Dummy variable, 0 for 1929-1946, 1 for 1947-1962.
E9	r_d	Average discount rate at all Federal Reserve Banks, per cent.
E10	R_e	Year-end ratio of member banks' excess to required reserves.
E11	G	Government expenditures, billions of 1954 dollars.
	E	Exports, billions of 1954 dollars.
E12	SD	Statistical discrepancy, billions of current dollars.
E13	I_R	Value of last twenty years investment.
E14	WB	Average weekly benefits for unemployed.

1 $Cd - a_1 Y = (.7 + a_2) Cd_{-1} - .7 a_1 Y_{-1} + a_3$

2 $Cn - a_4 Y = a_5 Cn_{-1} + a_6$

3 $R - a_{11} Y = a_{12} r_{-1} + a_{13} R_{-1} + a_{14}$

4 $(1 + a_{15})H - a_{15} X = (a_{15} + a_{16}) H_{-1} + a_{17}$

5 $Im - a_{18} X + a_{19} p = a_{19} Pm + a_{20} Im_{-1} + a_{21}$

6 $X - a_{22} R - a_{23} \frac{Nw}{9} - a_{24} h = Wg + .95(X - Wg)_{-1} + a_{22} I$

$- a_{23} Ng - Ns + .95(Nw - Ng + Ns)_{-1} - .95 a_{24} h_{-1} + a_{25}$

7 $h - a_{26} w + a_{27} Nw = - a_{26} w_{-1} + a_{27} (N1 - Ns) + a_{28}$

8 $W - a_{29} X = (1 - a_{29}) Wg + a_{30} (W - Wg)_{-1} + a_{31}$

9 $w + a_{32} Nw = w_{-1} + a_{32}(N1 - Ns) + a_{33} (P_{-1} - P_{-2}) + a_{34}$

10 $pSc + IVA - a_{38} (pPc + IVA - Tc) = a_{39} (pPc - Tc - pSc)_{-1} + a_{40}$

11 $p\Pi - pPc - IVA - a_{41} p X = a_{42} (p(\Pi - Pc) - IVA)_{-1} + a_{43}$

12 $p\Pi r - a_{44} pR - a_{44} pI = a_{45} (r - r_{-1}) + a_{46} (p\Pi r)_{-1} + a_{47}$

13 $X - Cd - Cn - R - H + Im = H_{-1} + I + \frac{G + E}{E_{11}}$

14 $pY - pX + pSc + IT + Tc + PT + SI - GT + IVA = GS - SD - D + ID$

15 $p\Pi - pX + pW + p\Pi r + IT + BT = - D + GS - SD$

16 $IT - C_2 pX = C_1$

17 $Tc - C_4\ pPc = C_3$

18 $(1 - C_6)\ PT - C_6\ pY = C_5$

19 $SI - C_8\ (pY + PT) = C_7$

20 $BT - a_{57}\ pX = a_{58}\ BT_{-1} + a_{59}$

21 $GT + Cg\ Nw = Cg\ (Nl - Ns) + C_{10}\ WB + C_{11}$

22 $pW - wh\ Nw = 0$

23 $IVA - a_{55}\ p = - a_{55}\ p_{-1} + a_{56}$

24 $I = a_7\ (X - Wg)_{-1} + a_8\ r_{-1} + (.95 + a_9)\ I_{-1} + a_{10}$

25 $rs = a_{51}\ rd + a_{52}\ Re_{-1} + a_{53}\ Du\ \&\ a_{54}$

26 $D = a_{48}\ \Sigma p(I + R) + a_{49}\ Du + a_{50}$

27 $r = a_{35}\ rs + a_{36}\ r_{-1} + a_{37}$

III. THE RESULTS OF THE SOLUTION

There are three ways in which the Gauss-Seidel process can be varied: (1) the normalization can be changed, that is, different equations can be solved for different variables; (2) the order of the equations can be changed; and (3) the starting estimate can be varied. From the mathematics of this method as applied to linear systems, it is clear that the normalization is critical; a system will converge for some normalizations and not for others. However, in none of the books on numerical analysis is there any mention of whether the ordering is a criterion for convergence. The model was normalized in approximately ten different ways. There is only a limited number of normalizations possible. Since each variable

must be associated with an equation, and the matrix is so sparse that there is not more than fifteen ways in which it can be renormalized. Of the ten normalizations tried, only two were found to converge. When the model was run with the original ordering it was found that variable six was consistently the first to diverge, eventually pulling the other variables with it. The model converged when equation nine was solved for variable six, equation seven was then solved for variable nine and equation six for variable seven. With this normalization approximately eight of the twenty-seven orderings converged. The orderings tried were generally choosen randomly. This result establishes that in some cases the ordering is very critical. The other ordering which converged was solving equation nine for variable seven and equation seven for variable nine. Thus the two successful normalizationsare very similar. However, if only slight modifications are made in these orderings, the system will diverge. The same orderings were used for each normalization, and for this normalization, twenty-five of the twenty-seven orderings tried converged.

It is interesting to note that the rate of convergence for the different orderings was by no means uniform. Some of the orderings converged in as few as nine iterations while others required well over a hundred. The criterion for convergence was rather loose. If none of the solution values changed by more than .01 in two successive iterations, the process was said to have converged. The initial estimates used were the actual values for the time period, but the process also converged with the zero vector as the initial estimate.

As the Gauss-Seidel process converges in each iteration it will produce a change in the current estimate of each variable. A damping factor was applied to this change to determine whether it was possible to force

convergence upon normalizations and orderings which normally diverge. With a damping factor the current estimate of the solution is given by

$$Y' = Y_{k-1} + d(Y_k - Y_{k-1})$$

where Y' is the new estimate of the vector of solutions, "d" is the damping factor, Y_{k-1} is the estimates solution from the last iteration, and Y_k is the solution which the Gauss-Seidel process would produce without a damping factor. When the damping factor is 1. this method becomes equivalent to the normal Gauss-Seidel process. Damping factors of .3, .5, and .7 were tried on all combinations of normalizations and orderings. In all of the normalizations for which the process had previously diverged, it also diverged with the damping factor. Thus if the normalization is one which diverges, applying a damping factor will be of no benefit. Yet for the two normalization which had converged, the applic-cation of a damping factor produced convergence in every case. Even for the orderings for which the normal step diverged, the process converged for all three damping factors. Thus it appears that at least for this model, the ordering of the equations is not critical if a damping factor is used. If this is true in general, it in effect removes one of the degrees of freedom from the Gauss-Seidel process. If the ordering is not critical for convergence, then one must only be concerned with the normalization and the original estimate.

There is in general little lost in speed when using a damping factor. On the contrary, the application of a damping factor often reduced the number of iterations necessary for convergence. This was particularly true when the normal step took many iterations to converge. If the normal

step did not converge rapidly it was usually because the estimates were oscillating around the true solution. By applying a damping factor in this situation the oscillations are damped out, and consequently requires less iterations to converge. However, when the normal step converged in less than twenty, the application of a damping factor only prolonged convergence. In all cases the damping factor of .7 provided more rapid convergence than the damping factor of.5, and similarly the damping factor of .5 was more rapid than .3. In all cases the process came to the identical solution. The application of a damping factor never forced the process to a false solution.

The mathematics implies that convergence can be increased by taking larger steps than the Gauss-Seidel method prescribes. Different acceleration factors were applied in a similar manner as the damping factors. A value of 1.1 was the smallest acceleration factor tried which produced convergence. Often when the normal step converged, the application of the acceleration factor of 1.1 made the process diverge. In no case did the acceleration factor produce more rapid convergence. Thus as far as this model is concerned, the acceleration factor never improved convergence. If one were to plot the number of iterations necessary for convergence as a function of the damping or acceleration factor for each normalization and each ordering, the function in all cases would be "U" shaped. For small damping factors the number of iterations is large. As the damping factor increases the number decreases, and then sharply increases. The optimum value of the damping function varies as a function of the normalization and the ordering, but it was never found to be greater than 1.

The variables which were renormalized to produce convergence were the variables for the index of hours worked per week, the index of hourly wages, and the number of wage and salary workers. The two normalizations which produced convergence also made the most economical sense. The difference between the two successful normalizations were not great in terms of economics. For the equations which were renormalized, there was no distinct dependent variable. Both of the successful normalizations made economic sense. However, if a slight modification was made in this normalization the process would diverge immediately. For instance, solving equation five for variable thirteen and equation thirteen for variable five made the process diverge. But this normalization made little economic sense, for it explained gross national product as a function solely of imports. Thus for this model the only normalizations converged were the normalizations which made economic sense.

From the results of solving this model a few generalizations can be made about an algorithm for producing convergence with the Gauss-Seidel iterative method. First the system should be normalized in the manner which makes the most economic sense. Any ordering may then be choosen, but a damping factor of approximately .7 should be used. If the process does not converge then a new normalization should be tried. If a normalization does not converge, it diverges immediately, so that it is immediately obvious whether the normalization produces convergence. Without using a damping factor one would have to attempt many orderings before it would be clear that it was the normalization and not the ordering which was causing the process to diverge.

If the model is to be solved many times, it is profitable to search

for an efficient ordering, since the rate of convergence between different orderings with a given normalization is great. The minimum number of iterations which the optimum ordering required to converge was the same for both normalizations. This implies that once a normalization has been found which converges it is not necessary to search for other normalizations in order to speed up convergence. The rate of convergence is a function chiefly of the ordering. There is a pay off between the amount of time spend searching for a optimum ordering and the amount of time which this ordering will save. The more times the model is to be solved, the more profitable it is to search for an optimum ordering. Eventually there may be an algorithm for optimum ordering, but there does not seem to be any correlation between the optimum orderings found in the solution of this model.

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